Linear Algebra Review

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1. Vector Space

- Defined over a field K ($\mathbb{R}$ or $\mathbb{C}$)
- Vectors $\in V$
- scalars $\in K$
- Operations

$$a\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} a \cdot v_1 + w_1 \\ a\, v_2 + w_2 \\ a\, v_3 + w_3 \end{bmatrix}$$

0 vector

$$\forall \psi \quad |\psi\rangle + 0 = |\psi\rangle$$

$$|\psi\rangle = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\left.\begin{array}{ll} |\ \rangle & \text{ket} \\ \langle\ | & \text{Bra} \end{array}\right\} \begin{array}{l} \text{Dirac Notation} \\ \text{Braket Notation.} \end{array}$$

2. Dirac Notation

Notation	Description
z^*	Complex conjugate of the complex number z.
	$(1+i)^* = 1 - i$ $\quad i = \sqrt{-1}$
$\vert\psi\rangle$ Column	Vector. Also known as a *ket*.
$\langle\psi\vert$ Row	Vector dual to $\vert\psi\rangle$. Also known as a *bra*.
$\langle\varphi\vert\psi\rangle$	Inner product between the vectors $\vert\varphi\rangle$ and $\vert\psi\rangle$.
$\vert\varphi\rangle \otimes \vert\psi\rangle$	Tensor product of $\vert\varphi\rangle$ and $\vert\psi\rangle$.
$\vert\varphi\psi\rangle$ $\vert\varphi\rangle\vert\psi\rangle$	Abbreviated notation for tensor product of $\vert\varphi\rangle$ and $\vert\psi\rangle$.
A^*	Complex conjugate of the A matrix.
A^T	Transpose of the A matrix.
$A^\dagger$	Hermitian conjugate or adjoint of the A matrix, $A^\dagger = (A^T)^*$.
	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}$.
$\langle\varphi\vert A\vert\psi\rangle$	Inner product between $\vert\varphi\rangle$ and $A\vert\psi\rangle$.
	Equivalently, inner product between $A^\dagger\vert\varphi\rangle$ and $\vert\psi\rangle$.

3. Bases and linear independence

- Spaning Set

$\{|v_1\rangle, \dots, |v_n\rangle\} \quad \forall |v\rangle \in V \qquad |v\rangle = \sum_i a_i |v_i\rangle$

$V = \mathbb{C}^2$

$|0\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$|v\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a_1|0\rangle + a_2|1\rangle$

$|v_1\rangle \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad |v_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$|v\rangle = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\frac{x_1 + x_2}{\sqrt{2}}}_{a_1} |v_1\rangle + \underbrace{\frac{x_1 - x_2}{\sqrt{2}}}_{a_2} |v_2\rangle$

- Linear dependent set

$\{|v_1\rangle, \dots, |v_n\rangle\} \quad \exists a_1 \dots a_n \qquad a_1|v_1\rangle + \dots + a_n|v_n\rangle = 0$

at least one $a_i \neq 0$

- Basis : a non-linear-dependent set that spans V

\# elements of basis = Dimension of V

4. Linear operators and matrices

- Linear operator V, W vector spaces

$$A : V \longrightarrow W \qquad A\left(\sum_i a_i |v_i\rangle\right) = \sum_i a_i A(v_i)$$

$$A|v_i\rangle = A(|v_i\rangle)$$

Identity operator $I_V |v\rangle = |v\rangle$

- Composition $A : V \to W \quad B : W \to X$

$$(BA)(|v\rangle) := B(A(|v\rangle)) = BA|v\rangle$$

- Matrix representation

$A : V \longrightarrow W$

$\{|v_1\rangle \dots |v_n\rangle\}$ a basis of V

$\{|w_1\rangle \dots |w_m\rangle\}$ a basis of W

$$A(|v_i\rangle) = \sum_{j=1}^{m} A_{ij} |w_j\rangle \qquad |v\rangle = a_1|v_1\rangle \dots + a_n|v_n\rangle$$

$$A(|v\rangle) = \sum_{i=1}^{n} a_i A(|v_i\rangle) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i A_{ij} |w_j\rangle$$

5. Pauli Matrices

$$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

$$X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

6. Inner product

- Inner product $(-,-): V \times V \to \mathbb{C}$

$$|v\rangle, |w\rangle \mapsto (|v\rangle, |w\rangle)$$

1. Linear on the second argument $(|v\rangle, \sum_i \lambda_i |w_i\rangle) = \sum_i \lambda_i (|v\rangle, |w_i\rangle)$
2. $(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^*$
3. $(|v\rangle, |v\rangle) \geq 0 \qquad (|v\rangle, |v\rangle) = 0 \iff |v\rangle = 0$

$V = \mathbb{C}^n$

$((v_1 \dots v_n), (w_1, \dots w_n))$

$= \sum_i v_i^* w_i$

- Dual vector

$$|v\rangle \Rightarrow \langle v| : V \to \mathbb{C}$$
$$|w\rangle \to (|v\rangle, |w\rangle)$$

$(|v\rangle, |w\rangle) = \langle v|(|w\rangle) = \langle v|w\rangle$

- Inner product space $\equiv$ Hilbert space if the dimension is finite
- orthogonality, norm $\quad |v\rangle$ and $|w\rangle$ are normal if $\langle v|w\rangle = 0$

$\frac{|v\rangle}{\| |v\rangle \|}$ is normal $\quad |v\rangle$ is normal if $\| |v\rangle \| = \langle v|v\rangle = 1$

- Orthonormal Basis

a basis $\{|v_1\rangle \dots |v_n\rangle\}$ is orthonormal if $\forall_i \; \| |v_i\rangle \| = 1$, $\forall_{i \neq j} \; \langle v_i | v_j\rangle = 0 \iff \langle v_i|v_j\rangle = \delta_{ij}$

- Gram - schmidt

$\{|w_1\rangle, \dots, |w_d\rangle\}$ a basis

$$|v_1\rangle = \frac{|w_1\rangle}{\| |w_1\rangle \|} \qquad |v_{k+1}\rangle = \frac{|w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle |v_i\rangle}{\left\| |w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle |v_i\rangle \right\|}$$

$\{|v_1\rangle, \dots, |v_d\rangle\}$ is an orthonormal basis.

- Outer product

$|v\rangle \in V \quad |w\rangle \in W \quad |w\rangle\langle v| : V \longrightarrow W$

$$|u\rangle \longmapsto |w\rangle\langle v|u\rangle$$

$$|w\rangle\langle v| \left(|u\rangle \right) = |w\rangle\langle v|u\rangle = \langle v|u\rangle |w\rangle$$

$$|v\rangle\langle w|y\rangle\langle w|x\rangle$$

$$= \langle w|y\rangle\langle w|x\rangle |v\rangle$$

$\{|1\rangle, \dots, |d\rangle\}$ orthonormal Basis

$$O = \sum_{i=1}^{d} |i\rangle\langle i| = I \qquad O : V \to V$$

$$O(|v\rangle) = \sum_{i=1}^{d} |i\rangle\langle i|v\rangle = \sum_{i=1}^{d} v_i |i\rangle = |v\rangle$$

$|v\rangle = v_1|1\rangle + \dots + v_d|d\rangle$

$$\langle i|v\rangle = v_1 \cancel{\langle i|1\rangle}^{0} + \dots v_i \cancel{\langle i|i\rangle}^{1} \dots + v_d \cancel{\langle i|d\rangle}^{0}$$

$$= v_i$$

7. Eigenvectors and Eigenvalues (Sect. 2.1.5)

- Eigenvector

Eigenvalue

Eigenvector

$$A|v\rangle = \lambda|v\rangle$$

- characteristic function

$$c(\lambda) \equiv \det|A - \lambda I| \qquad c(\lambda) = 0$$

- Diagonal representation

$$A = \sum_i \lambda_i |i\rangle\langle i|$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$A \equiv \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 2|0\rangle\langle 0| + 2|1\rangle\langle 1|$$

8. Adjoints and Hermitian operators

- Adjoint

$$(|v\rangle, A|w\rangle) = (\overset{\text{Adjoint of } A}{A^\dagger}|v\rangle, |w\rangle)$$

$$(AB)^\dagger = B^\dagger A^\dagger$$

A is matrix $\qquad A^\dagger = (\overset{\text{Conjugate}}{A^*})^T$

- Hermitian or Self adjoint

$$A = A^\dagger$$

- Projector

W is a subspace of V

$|1\rangle, \dots |d\rangle$ orthonormal basis for V, $|1\rangle \dots |k\rangle$ o.b. for W.

$$P = \sum_{i=1}^{k} |i\rangle\langle i| \qquad |v\rangle \in V \quad P|v\rangle \in W$$

- Normal $\quad AA^\dagger = A^\dagger A$

- Unitary $\quad U^\dagger U = I \qquad UU^\dagger = I$

$$(U|v\rangle, U|w\rangle) = \langle v|U^\dagger U|w\rangle$$
$$= \langle v|I|w\rangle$$
$$= \langle v|w\rangle$$

9. Tensor Product

- Tensor product of spaces

V m-dimensional with basis $\{|v_i\rangle\}_{i=1..m}$

W n-dimensional. with basis $\{|w_j\rangle\}_{j=1..n}$

$V \otimes W$ mn-dimensional. with basis $\{|v_i\rangle \otimes |w_j\rangle\}_{\substack{i=1..m \\ j=1..n}}$

- Tensor product properties

(1) For an arbitrary scalar z and elements $|v\rangle$ of V and $|w\rangle$ of W,

$$z\,(|v\rangle \otimes |w\rangle) = (z|v\rangle) \otimes |w\rangle = |v\rangle \otimes (z|w\rangle)\,. \qquad (2.42)$$

(2) For arbitrary $|v_1\rangle$ and $|v_2\rangle$ in V and $|w\rangle$ in W,

$$(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle. \qquad (2.43)$$

(3) For arbitrary $|v\rangle$ in V and $|w_1\rangle$ and $|w_2\rangle$ in W,

$$|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle. \qquad (2.44)$$

$|v\rangle \otimes |w\rangle$

$\equiv |v, w\rangle \equiv |vw\rangle$

- Linear operators on Tensor product spaces

A an operator in V, B an operator on W

$$(A \otimes B)\,(|v\rangle \otimes |w\rangle) \equiv A|v\rangle \otimes B|w\rangle$$

- Kronecker Product

$$\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \otimes \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \\ \vdots \\ v_m \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \end{bmatrix} = \begin{bmatrix} v_1 w_1 \\ v_1 w_2 \\ \vdots \\ v_1 w_n \\ v_2 w_1 \\ \vdots \\ v_2 w_n \\ \vdots \\ v_m w_1 \\ \vdots \\ v_m w_n \end{bmatrix}$$

$$\underline{|01\rangle} = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

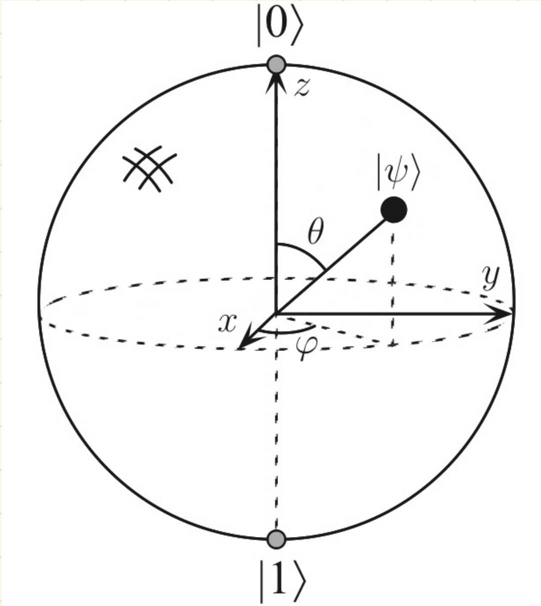
$$|v\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \qquad |v\rangle = v_1 |00\rangle + v_2 |01\rangle + v_3 |10\rangle + v_4 |11\rangle$$

$$|v\rangle = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}$$

$$\alpha |01\rangle + \beta |10\rangle$$

$$|\alpha| + |\beta| = 1$$

10. Qubits and the Bloch Sphere (Sect 1.2)



$$A : V \longrightarrow W \qquad \begin{matrix}\{V_1 \dots V_m\} \longrightarrow \{|V_1\rangle, |V_2\rangle \dots |V_m\rangle\} \\ \{w_1 \dots w_n\}\end{matrix}$$

$$A|v_j\rangle = \sum_i \underset{W\;\;V}{A_{ij}}\, |w_i\rangle$$

$$A_{23} = \langle w_2 | A(|v_3\rangle)$$

$$= \langle w_2 | A | v_3 \rangle$$

$$\langle w_2 | \overbrace{A | V_3\rangle}$$

$$= \langle w_2 \left(\sum_i A_{i3} |w_i\rangle \right)$$

$$= \sum_i A_{i3} \langle w_2 | w_i \rangle$$

$$= A_{23} \langle w_2 | w_2 \rangle$$

$$= A_{23}.$$

$$A = |v_j\rangle\langle v_k|$$

$$A_{lm} = \langle v_l | \overbrace{v_j\rangle\langle v_k}^{A} | v_m \rangle$$

$$A_{lm} = \begin{cases} 1 & \text{if } l = j \text{ and } m = k \\ 0 & \text{o.c.} \end{cases}$$